

Antiparticles in new quantum mechanics for a relativistic bosonic particle

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We construct a relativistic quantum mechanics for a boson which sustains the positive definite particle probability for the relativistic boson. In this theory, we settle down the well-known 85 years old unsolved problem on the particle probability of the Klein-Gordon quantum mechanics. The solutions for a boson obtained from our theory possess a negative solution as well as a positive one, similar to those of the Dirac quantum mechanics for a fermion. We observe in our theory that the Lorentz rotations do not produce a fermionic characteristic related to the 4π -rotation, and they do not change the phase factor of the wavefunction of the bosonic particle at all. In the Lorentz boosts, we find aspects similar to a relativistic fermionic particle. Moreover, we construct a vacuum state with all fermion and boson negative energy levels filled and all fermion and boson positive energy levels empty, so that we can have unification of the bosonic sea and the Dirac fermionic one. In this unified sea we investigate the bosonic and fermionic antiparticles associated with the unified vacuum on an equal footing, and we obtain a charge conjugation operator to study decay modes of the particles and antiparticles.

I. INTRODUCTION

Since Dirac performed the first theoretical prediction of the positron, antiparticle of the electron in 1928 [1], there have been lots of progresses in this antiparticle and even antimatter area, theoretically and experimentally. The prediction of the positron was experimentally confirmed by Anderson in 1933 [2]. Recently, the STAR collaboration reported the observation of antihypertritons consisting of an antiproton, an antineutron and an antilambda hyperon, which were produced by colliding gold nuclei at high energy [3]. The STAR collaboration also observed the antimatter helium-4 nucleus in high-energy nuclear collisions [4]. The ALPHA collaboration reported that they observed confined antihydrogen for 10^3 seconds at CERN [5]. Their calculations indicated that most of the trapped anti-atoms approach the ground state. Moreover, an experimental evidence of the reflection of low energy antiprotons by an aluminum wall was reported [6].

On the other hand, it is well known that the unification of the quantum mechanics [7, 8] and general relativity [9] is difficult. Specifically, even bosonic particle quantum mechanics and special relativity have not been unified until now [10–12]. For instance, the Klein-Gordon relativistic quantum mechanics [10–12] for a bosonic particle is known to have a pathological aspect that the particle probability is *not* positive definite. However, the relativistic quantum mechanics possessing the positive definite particle probability has been successfully constructed for a fermionic particle by Dirac [1].

To fix this no-positive definite particle probability problem in a bosonic particle, one introduces the quantum field theoretical approaches [13]. The quantum field theory [13] however remains unsolved in the sense that in this formalism we need many topological types of Feynman diagrams and lots of loop corrections associated with these diagrams [14], for instance. It is one of the reasons that (super)string theory [15, 16] has been formulated and still the research in this direction is in progress.

In this paper, in order to solve the unsettled problem on the boson particle probability, we will construct a proper quantization formalism for a relativistic bosonic particle possessing *positive definite particle probability*, similarly to the relativistic quantum mechanics for a fermionic particle. The advantage of this relativistic quantum mechanics for a boson is a unification of the newly discovered bosonic sea and the fermionic Dirac sea, in which we can investigate the bosonic and fermionic antiparticles associated with the unified sea on an equal footing.

This paper is organized as follows. In Section II, we will introduce the relativistic quantum mechanics for a bosonic particle which describes equations of motion for the relativistic boson. We will then construct explicitly the continuity equation and the solution to relativistic bosonic equations, to discuss the positive definiteness in particle probability. In Section III, we will study the Lorentz transformations including the rotations and boosts to see the bosonic properties of the wave functions of the particle. In Section IV, we will investigate the charge conjugation, antiparticles and their decay modes in our theory.

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II. RELATIVISTIC QUANTUM MECHANICS FOR A BOSONIC PARTICLE

We start with equation for relativistic quantum mechanics for a bosonic particle, $H\phi = E\phi$ with

$$H = \vec{\alpha} \cdot \vec{p} + \beta m, \quad (2.1)$$

where α_i ($i = 1, 2, 3$) and β are 2×2 matrices. Using the following simple algebra

$$E^2 = m^2 + \vec{p}^2, \quad (2.2)$$

we obtain relations between α_i and β ,

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}, \quad \beta^2 = I, \quad \{\alpha_i, \beta\} = 0, \quad \alpha_i^\dagger = \alpha_i, \quad \beta^\dagger = \beta, \quad (2.3)$$

where I is a 2×2 unit matrix. Here we note that eigenvalues of α_i or β are ± 1 and $\text{tr } \alpha_i = \text{tr } \beta = 0$. Moreover, for bosonic representation, the minimal dimension is two. We note that this situation is different from Dirac case of the fermionic relativistic quantum mechanics.

As in the Dirac relativistic formalism with the positive definiteness in fermion probability, exploiting the above relations we construct a relativistic quantum mechanics in which an equation for a bosonic particle is given as follows

$$(\vec{\alpha} \cdot \vec{p} + \beta m)\phi = E\phi, \quad (2.4)$$

where $\phi = (\phi_1, \phi_2)^T$ and

$$\vec{\alpha} = \hat{p}\sigma_1, \quad \beta = \sigma_3 \quad (2.5)$$

with σ_1 and σ_3 being the first and third Pauli matrices, respectively. In this relativistic quantization scheme of bosonic particle, we emphasize the characteristics that the matrices $\vec{\alpha}$ are associated with the unit vector of the particle linear momentum \vec{p} .

We now arrive at the relativistic quantum mechanical equation for a relativistic bosonic particle, by modifying (2.4) as follows¹

$$(i\gamma^\mu \partial_\mu - m)\phi = 0, \quad (2.6)$$

where

$$\gamma_0 = \beta = \sigma_3, \quad \vec{\gamma} = \beta \vec{\alpha} = \hat{p}i\sigma_2. \quad (2.7)$$

Here we note that we have found the equation (2.6) of first order in the time derivative, which allows a straight forward probability interpretation as in the Schrödinger equation [7, 8], as shown below.

Taking Hermitian conjugate of (2.6), which is given by $\bar{\phi}(i\gamma^\mu \overleftarrow{\partial}_\mu + m) = 0$ and $\bar{\phi} = \phi^\dagger \gamma^0$, we find the continuity equation as follows

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0, \quad (2.8)$$

where

$$\begin{aligned} \rho &= \bar{\phi} \gamma^0 \phi = \phi_1^* \phi_1 + \phi_2^* \phi_2, \\ \vec{J} &= \bar{\phi} \vec{\gamma} \phi = \hat{p}(\phi_1^* \phi_2 + \phi_2^* \phi_1). \end{aligned} \quad (2.9)$$

One notes that the particle probability density ρ is real and *positive definite*. The real vector \vec{J} is a current of the particle and it is parallel to \vec{p} of the particle, as shown in (2.9). We reemphasize that the quantities in the continuity equation (2.8) are physically well defined to yield a good quantization, differently from the Klein-Gordon continuity equation where the corresponding particle density ρ_{KG} for a relativistic bosonic particle is *not* positive definite [12]. This deficiency in the Klein-Gordon theory remains unsolved for 85 years and now it is finally settled down.

¹ Even though in this paper we have used α_i , β and γ^μ notations similar to the Dirac theory for relativistic fermionic particle, the differences between our theory and Dirac one are understood in the context.

We now consider the relativistic bosonic equations (2.4) or (2.6), which describes a relativistic motion of a particle and does not concern a many particle system. First, for the case of the rest particle with $\vec{p} = 0$, we have a solution of the form

$$\phi_+^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-imt}, \quad \phi_-^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{+imt}, \quad (2.10)$$

where the first and second wave functions describe a positive-energy solution and a negative-energy one, respectively. Second, for a solution for a relativistic bosonic particle with non-vanishing linear momentum \vec{p} , there are two types of solutions in this relativistic system as in the case of the above rest particle.

Since for the relativistic bosonic particle satisfying the relation (2.2) we have two kinds of solutions with $E = \pm(m^2 + \vec{p}^2)^{1/2}$ when $\vec{p} \neq 0$, we introduce an ansatz for a positive solution with an upper sign and a negative one with a lower sign,

$$\phi(x) = \phi(p) e^{\mp i p \cdot x}, \quad (2.11)$$

with $p \cdot x = p^\mu x_\mu = p_0 t - \vec{p} \cdot \vec{x}$. Inserting (2.11) into the relativistic boson equation (2.6), we obtain equations of motion of the form in the momentum space

$$(\not{p} \mp m) \phi(p) = 0, \quad (2.12)$$

where we have used $\not{p} = \gamma^\mu p_\mu$.

Exploiting the ansatz (2.11) and the corresponding equations (2.12), we find the positive energy solution with $E > 0$ given by

$$\phi_+(x) = u(p) e^{-i p \cdot x}, \quad (2.13)$$

where

$$u(p) = \left(\frac{E + m}{2m} \right)^{1/2} \begin{pmatrix} 1 \\ \frac{|\vec{p}|}{E + m} \end{pmatrix} e^{-i p \cdot x}. \quad (2.14)$$

We then have $\bar{u}u = 1$ and $u^\dagger u = \frac{E}{m}$.

Next we also find the negative energy solution with $E < 0$

$$\phi_-(x) = v(p) e^{+i p \cdot x}, \quad (2.15)$$

where

$$v(p) = \left(\frac{|E| + m}{2m} \right)^{1/2} \begin{pmatrix} \frac{|\vec{p}|}{|E| + m} \\ 1 \end{pmatrix} e^{+i p \cdot x}. \quad (2.16)$$

We then have the normalization relations $\bar{v}v = -1$ and $v^\dagger v = \frac{|E|}{m}$. Additionally, we obtain the other normalization conditions $\bar{u}v = 0$ and $\bar{v}u = 0$. Here one notes that in the vanishing \vec{p} limit, the positive and negative energy solutions $\phi_\pm(x)$ in (2.13) and (2.15) are reduced to the corresponding solutions $\phi_\pm^0(x)$ in (2.10).

We note here that the above $u(p)$ and $v(p)$ satisfy

$$(\not{p} - m)u(p) = 0, \quad (\not{p} + m)v(p) = 0 \quad (2.17)$$

and their Hermitian conjugate equations are given by $\bar{u}(\not{p} - m) = 0$ and $\bar{v}(\not{p} + m) = 0$ where we have used the identity $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$. Here, as in the Dirac theory, we interpret the negative energy solution to describe an antiparticle associated with the particle given by the positive energy solution. More details will be given later.

III. LORENTZ TRANSFORMATIONS

In order to consider the Lorentz transformation, we should be careful in treating the relativistic quantum mechanics for the bosonic particle. Namely, the matrices $\vec{\alpha}$ are associated with the unit vector of the particle linear momentum \vec{p} . To proceed, we identify $\hat{p} = \hat{z}$ and we consider the transformations in coordinates

$$x'^\mu = a^\mu{}_\nu x^\nu \quad (3.1)$$

and those in derivatives

$$\partial'_\mu = a_\mu{}^\nu \partial_\nu. \quad (3.2)$$

Similar to the Dirac relativistic quantum mechanics, we consider wavefunction transformation as follows

$$\phi'(x') = S\phi(x), \quad (3.3)$$

where S is a 2×2 matrix to be fixed.

In this theory we consider covariance of the equation (2.6), by investigating the following equation

$$(i\gamma^\mu \partial'_\mu - m)\phi'(x') = 0, \quad (3.4)$$

which, exploiting (3.2) and (3.3), produces a covariance condition

$$S^{-1}\gamma_\mu a^\mu{}_\nu S = \gamma_\nu. \quad (3.5)$$

Here one notes that γ^μ has dependence of \vec{p} in this relativistic bosonic quantum mechanics.

Now, we consider a x - y rotation around z axis parallel to the particle direction \vec{p} , in which we have the relevant 2×2 matrix² $a^{\mu\nu}$ with $\mu, \nu = 1, 2$. In a small θ limit, $a^{\mu\nu}$ becomes

$$a^{\mu\nu} = g^{\mu\nu} + \epsilon^{\mu\nu}, \quad (3.6)$$

where $g^{\mu\nu} = \text{diag}(-1, -1)$ and

$$\epsilon^{\mu\nu} = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}. \quad (3.7)$$

We next introduce S of the form

$$S = I - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}. \quad (3.8)$$

After some algebra with considering the covariance in (3.5), for the rotation around z direction we find

$$S_{rot}^z = I. \quad (3.9)$$

The other rotations S_{rot}^x and S_{rot}^y around x and y directions, respectively, produce the same result (3.9). Here one notes that the eigenvalues of S_{rot}^x , S_{rot}^y and S_{rot}^z are all positive definite, with $+1$, so that we can not have a fermionic characteristic which is related to the negative eigenvalue -1 and is associated with the 4π rotation. Moreover, with this rotation at hand, the wavefunction of the bosonic particle does not change the phase factor at all.

Similarly, for a t - z boost along z axis parallel to the particle direction \vec{p} , as before we have the relevant 2×2 matrices $a^{\mu\nu}$ in terms of u . In a small u limit, $a^{\mu\nu}$ becomes the form (3.6) where $g^{\mu\nu} = \text{diag}(+1, -1)$ and

$$\epsilon^{\mu\nu} = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}. \quad (3.10)$$

After some algebra similar to the above rotation cases, for the finite t - x , t - y and t - z , boosts we obtain

$$S_{boost}^x = S_{boost}^y = I, \quad S_{boost}^z = e^{-u\sigma_1/2}. \quad (3.11)$$

Here one notes that these rotation and boost operators satisfy the following identities

$$S_{rot}^\dagger = \gamma^0 S_{rot}^{-1} \gamma^0, \quad S_{boost}^\dagger = \gamma^0 S_{boost}^{-1} \gamma^0. \quad (3.12)$$

² In this dimensionality, we keep the relevant indices after ignoring t and z components. This rule in the notations will be used in the forthcoming computations associated with the indices.

IV. ANTIPARTICLES AND DECAY MODES

Now we consider antiparticles and their decay modes in the relativistic quantum mechanics for a bosonic particle. As in the fermion Dirac sea [17], we will introduce a bosonic hole in bosonic sea to investigate the meaning of the negative solution in our relativistic quantum mechanics for a bosonic particle. We take a more general ansatz that a vacuum state can be regarded as one with both all fermion and boson negative energy levels filled and all fermion and boson positive energy levels empty.

To proceed, we include a minimal electromagnetic interaction in (2.6) to yield

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\phi = 0, \quad (4.1)$$

where e is charge of the particle. For instance, a hole in the bosonic π^- sea corresponding to the absence of an energy $-E$ ($E < 0$) and absence of a charge e ($e < 0$) is equal to the presence of a π^+ of positive energy $+E$ and charge $-e$.

One can now have a one-to-one correspondence between the negative energy solution of (4.1) for π^- and the π^+ eigenfunction ϕ_c which will be a positive energy solution of the following equation,

$$(i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu - m)\phi_c = 0. \quad (4.2)$$

Here one notes that the π^- then comes out from the bosonic hole interpretation as the absence of the negative energy solution of (4.2). After some manipulation, we have the relation between the π^- wavefunction ϕ in (4.1) and the π^+ wavefunction ϕ_c in (4.2) as follows

$$\phi_c = C\gamma_0\phi^*, \quad (4.3)$$

where $C\gamma_0$ is a charge conjugation operator. Here one notes that C is constrained by the following identity

$$C^{-1}\gamma^\mu C = -\gamma^{\mu T}, \quad (4.4)$$

and in our theory it is given by

$$C = -\sigma_1. \quad (4.5)$$

Combining (4.3) and (4.5) and for instance applying the result to the rest boson case, we can readily find that the negative solution ϕ_{c-}^0 is equal to ϕ_+^0 in (2.10). Namely, the absence of a negative energy π^- is equal to the presence of a positive energy π^+ at rest. This statement can be applied to more general cases without loss of generality. The positive energy solution thus delineates the particle π^- of mass m and charge e ($e < 0$), while the negative energy solution describes the antiparticle π^+ of mass m and charge $-e$.

Next, we consider the case of electron and positron experimental decay mode via electromagnetic interaction,

$$e^- + e^+ \rightarrow \gamma + \gamma, \quad (4.6)$$

where we exploit the negative energy solution in the Dirac fermion sea to specify the positron theoretically. For the case of charged pions π^- and π^+ , we cannot have a bosonic decay mode experimentally

$$\pi^- + \pi^+ \nrightarrow \gamma + \gamma. \quad (4.7)$$

Phenomenologically, the bosonic decay mode (4.7) is forbidden due to the absence of electromagnetic interaction directly coupled to the charged pions [12], contrast to the fermionic decay mode (4.6). Instead of the decay mode (4.7), we have dominantly favorable decay modes for charged pions via weak interaction [18, 19]

$$\pi^- \rightarrow \mu^- + \bar{\nu}_\mu, \quad \pi^+ \rightarrow \mu^+ + \nu_\mu. \quad (4.8)$$

Moreover, we observe that the neutral pion π^0 is equal to its antiparticle, namely the neutral pion itself which is consistent with the conventional quark model, where the quark combination of the neutral pion is the same as that of the bosonic antiparticle of the neutral pion. It is interesting to see that it is dominantly favorable to have a main decay mode [18, 19],

$$\pi^0 \rightarrow \gamma + \gamma. \quad (4.9)$$

This decay mode can be understood in terms of π^0 particle and/or π^0 antiparticle annihilating into two photons in an electromagnetic interaction process associated with the photons, since π^0 is both a particle and an antiparticle

itself as shown before. Next we consider second common Delitz decay mode of π^0 into a photon and electron positron pair [19, 20],

$$\pi^0 \rightarrow \gamma + e^- + e^+. \quad (4.10)$$

In this decay mode we have bosonic (anti)particles π^0 and γ and fermionic particles e^- and e^+ involved. In our unified sea, the bosonic (anti)particle π^0 and the fermionic antiparticle e^+ participate in this process together.

Recent experimental results on antimatter at CERN show that confined antihydrogen lives for 10^3 seconds [5]. The RHIC experiment observed the antimatter helium-4 nucleus in high-energy nuclear collisions [4]. Since the decay mode (4.9) is known to exist experimentally, it will be a good challenge to retest the decay mode (4.7) in a modern accelerator such as the RHIC and the LHC with very high energy facility.

The relativistic quantum mechanical equations (2.4) and/or (2.6) can be rewritten as

$$(\square + m^2)\phi_a = 0, \quad (a = 1, 2). \quad (4.11)$$

We note that the above equations are not the Klein-Gordon equation since we have the *two* components ϕ_1 and ϕ_2 in (4.11). Moreover, the particle number density in our quantization was shown to be positive definite, different from the Klein-Gordon equation possessing ill defined particle probability density explained before.

In particular, in the massless limit, the relativistic quantum equation (4.11) for a bosonic particle is reduced to the photon case as follows

$$\square\phi_a = 0, \quad (a = 1, 2), \quad (4.12)$$

which describes the propagation of the photon with $\phi_1 = \phi_2$. One notes that there is no difference between particle and antiparticle in this photon case.

In summary, we have studied the properties of the *quantum mechanics* for a *relativistic boson*. In this scheme, we have described the equations of motion for the relativistic boson, and we have also defined particle probability density successfully, contrast to the Klein-Gordon relativistic quantum mechanics. In order to discuss the positive definiteness in particle probability, we have constructed explicitly the continuity equation and the solutions to the relativistic bosonic equations. As in the Dirac theory, we have also interpreted the negative energy solution to describe an antiparticle associated with the particle given by the positive energy solution. Under the Lorentz transformation with the rotation, we cannot have a fermionic characteristic associated with the so called 4π rotation in the Dirac theory, as expected. Moreover, the wavefunction of the bosonic particle under this rotation does not change phase factor. In the case of the boost we have aspects similar to a relativistic fermionic particle. We have then investigate antiparticles and their decay modes in the framework of the relativistic quantum mechanics for a bosonic particle. We have taken a more general ansatz that a vacuum state can be unified to be one with all fermion and boson negative energy levels filled and all fermion and boson positive energy levels empty.

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